The Analysis of Certain Curves by Means of Derived Local Separating Points

A Dissertation

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

G. E. SCHWEIGERT

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THE ANALYSIS OF CERTAIN CURVES BY MEANS OF DERIVED LOCAL SEPARATING POINTS.

By G. E. SCHWEIGERT.

1. *Introduction.* It is well known [1] that the removal of the local separating points [2] from a regular curve leaves a set of dimension zero. This suggests that a similar result may be true for an hereditarily locally connected continuum, that is, a continuum having the property that every subcontinuum is locally connected. However an example by Gehman [3] shows that non-degenerate components may remain in such a continuum after the removal of its local separating points. In this example a line segment is the only component that remains and if we remove in addition the local separating points of the line segment, the result is a set of dimension zero. Thus although it is clearly necessary to remove points other than the local separating points, the situation in this example suggests that *given an hereditarily locally connected continuum, a set of dimension zero may be reached by a countable iteration of the process of removing the local separating points from the components that remain at each successsive stage.* It is to this problem that sections 2, 3, 4, and 5 of this paper are devoted.

2. *Derived local separating points.* For any subset M of a compact, metric, hereditarily locally connected continuum H, let $L_0(M)$ denote the set of all local separating points of M. Then we may write $H - L_0(H) = \Sigma C_k^1 + P_1$ where the C_k^1 are the non-degenerate components in the set on the right and P_1 is a set of dimension zero. In doing this we make use of the fact [4] that an arbitrary subset of an hereditarily locally connected continuum is made up of non-degenerate components and a zero-dimensional set, with the former in the form of a (null) sequence. Now let $L_1(H) = \Sigma L_0(C_k^1)$, and proceed to represent the set $H - [L_0(H) + L_1(H)]$ in a similar fashion. This process

[1] G. T. Whyburn, "Über die Strukture regulärer Kurven," *Wiener Akademie Anzeiger* (1930), Nr. 6.

[2] If p is a cut-point of a connected neighborhood R in a connected and locally connected space then p is said to be a *local separating point*, that is, $R - p = R_1 + R_2$ where $\bar{R}_1 R_2 = 0 = R_1 \bar{R}_2$ and neither R_1 nor R_2 is vacuous.

[3] *Annals of Mathematics*, vol. 27 (1926), p. 43.

[4] G. T. Whyburn, "Concerning hereditarily locally connected continua," *American Journal of Mathematics*, vol. 53, no. 2 (1931), Corollary c, p. 379 and Theorem 4, p. 377.

continues under an inductive scheme of definition which uses the transfinite ordinal numbers. For this purpose we assume that the components C_N^γ have been defined as the components of the set $H - \sum_{\beta<\gamma} L_\beta(H)$ for every $\gamma < \alpha$, and also that $L_\gamma(H) = \Sigma L_0(C_N^\gamma)$ for every $\gamma < \alpha$. Then the general terms are $H - \sum_{\beta<\alpha} L_\beta(H) = \Sigma C_i^\alpha + P_\alpha$ and $L_\alpha(H) = \Sigma L_0(C_i^\alpha)$ where: (a) for each pair (i, α), $i = 1, 2, 3, \cdots$ and α transfinite, C_i^α is a component of $H - \sum_{\beta<\alpha} L_\beta(H)$; (b) P_α is a set of dimension zero.

For each i the set C_i^α is said to be a *derived component of index* α relative to H. Similarly the points in the set $L_\alpha(H)$ are the *derived local separating points of index* α relative to the continuum H. Under this notation H is denoted by C_1^0. Due to a theorem by Wilder [5] it is known that each derived component is a locally connected set.

If we make use of the fact that the derived local separating points are dense in each derived component [6] we may put the solution to our first problem in the form of the next proposition.

THEOREM. *If H is any compact, metric, hereditarily locally connected continuum there exists a number α of the first or second number class such that $L_\alpha(H) = 0$.*

If $\alpha(H)$ is the first such ordinal number α, then $\alpha(H)$ is said to be the *index* of the continuum H.

The above theorem [7] will be established with the aid of the work of Menger and Reschovsky on rational curves.

3. *Unilateral and bilateral limit points.* A generic member of the derived components of index α will be denoted by C^α. Whenever C^α, C^β, C^γ, etc., are used they are to be thought of as members of a monotone sequence which terminates with the component of highest index.

The symbol R denotes any fixed neighborhood in H having a countable boundary and of such a nature that it is in diameter less than that of any component with which it appears. The boundary of any such neighborhood is

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[5] R. L. Wilder, *Proceedings of the National Academy of Sciences*, vol. 15 (1929), p. 616.

[6] This result is an easy consequence of the fact that any isolated point of an irreducible cutting between two points of a connected and locally connected set M is a local separating point of M. For the theorem quoted here see G. T. Whyburn, *Monatshefte für Mathematik und Physik*, vol. 36 (1929).

[7] A more elementary proof using only the material now at hand together with the fact that H is a rational curve could be given at this point.

a countable cutting of H for which we consider only the particular separation $H - F(R) = R + (H - \bar{R})$. Similarly for the cutting $F(R)C^\alpha$ of C^α we use but one separation $C^\alpha - F(R)C^\alpha = C^\alpha R + C^\alpha(H - \bar{R})$.

If X is a cutting of a set M and the point x belongs to X, then x is said to be a *unilateral limit point of M with respect to the fixed separation* $M - X = M_1 + M_2$ provided x is a limit point of M_1 or M_2 but not of both these sets. If x is a limit point of both M_1 and M_2 then x is a *bilateral limit point of M with respect to the separation.* Since we are to be concerned only with cases in which there is a known separation (as in the paragraph above) the shorter terms unilateral and bilateral limit point *of M* will be used.

It can easily be shown that if x is a unilateral limit point of C^α then x is a unilateral limit point of C^β provided that $x \in F(R)C^\beta$ and $\beta > \alpha$. In regard to the bilateral limit points, what amounts to the converse is true, namely, if x is a bilateral limit point of C^α then x is a bilateral limit point of any component of smaller index.

The α-th derivative of the boundary of any neighborhood N will be denoted by $F(N)^\alpha$. A set E belonging to the cutting $F(R)C^\alpha$ of C^α will be said to be *of type B^α* provided there exists for each point $y \in E$ a number $\alpha(y) < \alpha$ such that y is a unilateral limit point of $C^{\alpha(y)}$. It follows that y is a unilateral limit point of all those derived components of index greater than $\alpha(y)$, including C^α. Another most useful result in this connection is that y is a unilateral limit point of $C^{\alpha-1}$ when α is an isolated number. If E is a set of type B^α we shall write $E = B^\alpha$. The only sets E to be used here are the particular type shown in the next paragraph.

We now prove a lemma from which the principal proposition of this paper is to be deduced. The lemma is stated in terms of the symbols just developed and its second part is a direct consequence of the first.

LEMMA. (1) $[F(R) - F(R)^\zeta]C^\zeta = B^\zeta$; (2) *If x is a bilateral limit point of C^ζ and $x \in [F(R) - F(R)^{\zeta+1}]C^\zeta$, then x is a derived local separating point of index ζ.*

Proof. When $\zeta = 1$ it must be shown in part 1 that there is *no* bilateral limit point x of C^0 (that is, of H) among those isolated points of $F(R)$ which are in C^1. In this connection we recall that the set $F(R)$ is a *countable* cutting of H. This allows us to obtain a separation of H by means of its bilateral limit points alone and from this stage it is but an easy step to show that x is a local separating point of H. Thus x is not a point of C^1 which is contrary to the hypothesis for the existence of that point.

The point x described in part 2 must lie in $F(R)^1 - F(R)^2$ because

it is a bilateral limit point of C^1 and therefore cannot be an isolated point in $F(R)$. But this means that x is an isolated point of X_0, where X_0 is the set of all bilateral limit points of C^1. Then by means of the same argument as was used above it follows that x is a point of $L_1(H)$.

Now for the purpose of induction we assume that the Lemma is true for $\zeta < \eta$ and show that it holds for $\zeta = \eta$.

If we assume the contrary of what is to be shown, we have that some point x is a bilateral limit point of C^ζ for all $\zeta < \eta$ where $x \in [F(R) - F(R)^\eta]C^\eta$. It follows that if η is a limit number, there exists a number $\lambda < \eta$ such that x non-$\in F(R)^\lambda$. But then $x \in [F(R) - F(R)^\lambda]C^\lambda$, and furthermore x is a bilateral limit point of C^λ because that property holds for C^ζ where $\zeta < \eta$. This is impossible in view of the fact that $[F(R) - F(R)^\lambda]C^\lambda = B^\lambda$ holds by assumption and hence no point of the set on the left is a bilateral limit point of C^λ. If on the other hand η is an isolated number, we know that x is a bilateral limit point of $C^{\eta-1}$. Moreover $x \in [F(R) - F(R)^\eta]C^{\eta-1}$, since $x \in C^\eta$. Then by part 2 for $\zeta = \eta - 1$, we find that $x \in L_{\eta-1}(H)$. This is impossible when $x \in C^\eta$.

It remains to be shown that if x is a bilateral limit point of C^η and $x \in [F(R) - F(R)^{\eta+1}]C^\eta$, then $x \in L_\eta(H)$. Certainly such a point x cannot belong to $F(R) - F(R)^\eta$ because of the results just above; hence $x \in F(R)^\eta - F(R)^{\eta+1}$. Now let X_0 be the set of all bilateral limit points of C^η. Then $X_0[F(R) - F(R)^\eta]C^\eta = 0$ because the last two factors form a set of type B^η. It follows that x is an isolated point of X_0 and hence by the argument used earlier that $x \in L_\eta(H)$. This completes the induction, that is, the proof of the Lemma.

4. *Relation between index and genus.* A point p of a set M is said to be of *genus* α *in* M [8] provided: (a) there exists arbitrarily small neighborhoods in M closing down on p in such a way that the α-th derivative of the boundary of each neighborhood is vacuous; (b) for some $\epsilon > 0$ and every $\beta < \alpha$ there exists *no* neighborhood in M which contains p, is of diameter less than ϵ and has the further property that the β-th derivative of the boundary is vacuous. The genus of the point p will be denoted by $g(p)$.

Suppose there is a point p in a derived component C^α such that $g(p) \leq \alpha$. Then since p is also in $C^{g(p)}$, there exists a neighborhood R in H such that R contains p and does not contain all points of $C^{g(p)}$. Moreover R can be so chosen that $F(R)^{g(p)} = 0$ and $F(R)$ is countable. It follows from the fact that $C^{g(p)}$ is connected that there exists a point x of $F(R)C^{g(p)}$ such that x is a bilateral limit point of $C^{g(p)}$. Thus $x \in [F(R) - F(R)^{g(p)}]C^{g(p)}$ because

[8] *Geschlecht*—definition due to Menger. See *Kurventheorie* (Teubner), p. 294.

$F(R)^{g(p)} = 0$. This is contrary to the Lemma in section 3, hence we have shown that the following proposition is true.

THEOREM. *If H is any compact, metric, hereditarily locally connected continuum and C^α is any one of its derived components of index α, then each point of C^α is of genus greater than α.*

5. *Solution to the problem stated in section* 1. It has been shown by Reschovsky [9] that there is associated with each rational curve K a number α of the first or second class such that the genus of each point of K is less than or equal to α. The smallest number $g(K)$ for which this statement holds is called the genus of the rational curve.[10]

This classification applies to the hereditarily locally connected continuum H for it has been established by G. T. Whyburn [11] that each such continuum is a rational curve. Thus it is an obvious corollary to the Theorem immediately above that $g(H) \geq \alpha(H)$. This means that $\alpha(H)$ is also of the first or second number class, consequently the Theorem stated in section 2 is proven and thus we have a positive solution to our original problem.

6. *Regular and rational bases.* A subset B of an arbitrary set M is said to be a *regular basis* for M provided that, (1) each regular point of M is contained in arbitrarily small neighborhoods N_ϵ in M which close down on p in such a way that $F(N_\epsilon)$ is finite and is contained in B, (2) when p is of order k, the neighborhoods N_ϵ can be chosen so that $F(N_\epsilon)$ consists of exactly k points of B. Similarly B is said to be a *rational basis* for M provided that each rational point of M is contained in arbitrarily small neighborhoods in M which close down on that point and have countable boundaries in B. Obviously a set of dimension zero remains on removing a rational (regular) basis from a rational (regular) curve.

In the paper by G. T. Whyburn referred to in the introduction it is shown that B will be a countable regular basis for a regular curve K provided that $B = T + D$, where T is the set of all local separating points of order greater than 2 in K and D is a countable set which is dense on each free arc in K. It is our purpose to establish a proposition which is analogous to this one in that it enables us to choose a rational basis for a curve H from its derived local separating points. The next Theorem is offered as a solution to this problem.

THEOREM. *Let H be any compact, metric, hereditarily locally connected*

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[9] Reschovsky, *Fundamenta Mathematica*, vol. 15 (1930), p. 18.

[10] *Loc. cit.*

[11] "Concerning hereditarily locally connected continua."

continuum and let B be any subset of H such that $B = \sum_{i,a} T_i^a + D$, where D is a countable set, and such that in regard to each derived component C_i^a: (a) *T_i^a is the set of all derived local separating points of order greater than two in that component and of index equal to that of the component,* (b) *D is dense on each free arc. Then B is a countable rational basis for H and the part of B which is contained in each derived component is a countable regular basis for that component.*

In the statement of this Theorem we again follow the convention that H is uniquely the derived component of index zero; hence when H is a regular curve, this Theorem reduces to the one quoted in the above paragraph. The proof of this Theorem is an almost direct result of the following Lemma.

LEMMA. *If C is any connected and locally connected set and B is any subset of C such that $B = T + D$, where T is the set of all local sepaarting points of order greater than two in C and D is a countable set which is dense on each free arc in C, then B is a regular basis for C.* Moreover B is a *countable* regular basis provided the nodules in C form a null sequence.

This Lemma follows from an argument based on the work of G. T. Whyburn.[12]

Since each derived component C_i^a of the curve H is locally connected we may apply the Lemma to each of these components and obtain a regular basis $B_i^a = T_i^a + D_i^a$. This basis is a countable set in each case because the nodules are certainly in a null sequence when the containing space is hereditarily locally connected. Let T and D denote the sets $\sum_{i,a} T_i^a$ and $\sum_{i,a} D_i^a$ respectively and denote their sum by B. From this approach it is obvious that the part of B that is contained in each derived component is a regular basis for that component. Moreover there are but a countable number of derived components hence B is a countable set. Thus in order to complete the proof of the Theorem we have to show that B is a rational basis.

It is first shown that the non-degenerate components of $C_i^a - B_i^a$ are exactly the components of $C_i^a - L_0(C_i^a) = \Sigma C_k^{a+1} + P_{a+1}$ after which it is immediate that $H - B$ is a set of dimension zero. For the purpose of comparing the two sets, let $C_i^a - B_i^a = \Sigma \Gamma_j^{a+1} + Q_{a+1}$ where the Γ_j^{a+1} are the non-degenerate components of the containing set and Q_{a+1} is a set of dimension zero. Now assume there exists a point p in some fixed component Γ_j^{a+1} such that p non-ϵ C_k^{a+1} for all k. Then p is either a point of $L_0(C_i^a) - B_i^a$ or a point of P_{a+1}. When p is in the former set, let $q \epsilon \Gamma_j^{a+1}$ be a point distinct

[12] See his papers "On the structure of connected and connected im kleinem point sets," *Transactions of the American Mathematical Society*, vol. 32 (1930), pp. 926-943, and "Über die Struckture regulärer Kurven," *Wiener Anzeiger* (1930), Nr. 6.

from p. Then there exists a neighborhood R in C_i^a such that R contains p but does not contain q and such that $F(R) = x, y \epsilon B_i^a$. (The point p is of order two by hypothesis.) However Γ_j^{a+1} is connected and consequently $F(R)$ contains a point z which is distinct from x and y. This is a contradiction. When p is in the latter set P_{a+1}, the point p is contained in arbitrarily small neighborhoods in $C_i^a - L_0(C_i^a)$ each of which does not contain all of that set and has a vacuous boundary. On the other hand we have just shown that Γ_j^{a+1} can contain no point of $L_0(C_i^a) - B_i^a$ hence $\Gamma_j^{a+1} - p$ is contained in a component C_k^{a+1} for some k. These two situations are inconsistent and bring us to the result that $H - B$ is a set of dimension zero. But it obviously follows that $(H - B) + p$, where p is any point of B, is also a set of dimension zero and consequently B is a rational basis. This completes the proof of the Theorem.

It is to be noted that in each case where a basis is mentioned in this section, that basis is prescribed, except for a certain freedom in the choice of the sets which are dense on the free arcs, by the set which contains it. Thus it may be said that each basis is a *natural* basis.

7. *Conclusion.* We are now in a position to demonstrate that any hereditarily locally connected continuum H is topologically contained in one which is of index 1. For this purpose we need the fact that there exists a locally connected Universal Curve in Three-space which contains a subset topologically equivalent to each compact, metric curve.[13] Now let H be any hereditarily locally connected continuum and let T be a homeomorphism throwing H into a subset $T(H)$ of the Universal Curve. Let $T(B) = b_1 + b_2 + b_3 + \cdots$ be the image under T of a countable rational basis B in H. Then add to $T(H)$ a line segment $L(1)$ of a length 1 having the point b_1 in common with $T(H)$. In general $L(n)$ is to be a member of a monotone sequence of disjoint line segments which meets $T(H)$ in just one point $b_n \epsilon T(B)$ and has the length $1/n$. The curve $T(H) + \Sigma L(n)$ will then be of index 1.

One can easily see that if the original curve H is contained in a space of a finite number of dimensions, it can be augmented in the manner described above and the resulting curve will have the same genus as the original curve. This suggests to the author that, while some examples of his (similar to the one by Gehman) showing plane curves with indices $2, 3, 4, \cdots, \omega$ are cases for which $\alpha(H) = g(H)$, this method will alter those examples to show that there are cases in which $\alpha(H)$ takes the full range allowed by the inequality $\alpha(H) \leq g(H)$. In contrast to this we know that if H is a regular curve $\alpha(H) = g(H) = 1$.

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[13] Menger, *Kurventheorie*, p. 365.

VITA.

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